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Quantum statistical properties of a quantum theory of optical bistability

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The strictly quantum driven Dicke model of N two-level atoms on the same site shows conventional optical bistability if and only if cavity feedback is included. In this model we find that for $N \to \infty$ the photon statistics of the transmitted field are Poisson on the lower branch of the output–input curve, but on the upper branch approach Bose–Einstein in the hysteresis region, with reversion to Poisson only for very strong input fields.

The device potential of optical bistability (o.b.) is now so plain that it may be salutary to recall that the phenomenon also has fundamental interest. Within a semiclassical (decorrelation) approximation it can be exhibited as a fine example of a phase transition far from equilibrium (see, for example, Agarwal *et al.* 1978). So studies of any quantum features are important and one way of studying them is through the photon statistics of the different branches of the transmitted field. Unfortunately even the model problem of the o.b. of two-level atoms in an extended cavity is not solved as a quantum theory. So nothing can be said about the photon statistics.

However we have solved the quantum problem of the steady state of N two-level atoms on a single site (a Dicke model) driven by a coherent state continuous wave (c.w.) laser field completely (see, for example, Hassan *et al.* 1984). We have calculated the intensity correlations of the fluorescent radiation $g^{(n)}(0) \equiv \langle S_+^n S_-^n \rangle / \{\langle S_+ S_- \rangle\}^n$, where $S_\pm$ are collective atomic operators: $g^{(n)}(0) = 1$ below a certain threshold but, for example, $g^{(2)}(0) \to 1.2$ (when $N \to \infty$) above it. So there is a second-order type phase transition from coherence to partial coherence at the threshold (cf. Hassan *et al.* 1980, 1984 and references therein). Although cavity feedback is no longer viewed as essential for o.b. (see, for example, the papers by Garmire *et al.* (this symposium) this quantum model shows conventional o.b. if, but only if, the model is coupled to a cavity (Puri *et al.* 1984). Moreover the quantum character is essential; the o.b. disappears if the theory is decorrelated. Figure 1 illustrates the results: as $N \to \infty$ a cusp develops at the switch-up point ($x = 1$) and the second-order type phase transition occurs there.

Although the Dicke model seems unphysical, results for the model coupled to a single cavity mode of black body radiation are in remarkable agreement with recent observations on high rydberg atoms (Raimond *et al.* 1982*a*, *b*; Hildred *et al.* 1984). This suggests similar agreement will be possible for coherent light. A way of investigating the o.b. (rather than the second-order phase transition) is through the statistics of the transmitted light. This paper sketches the theory. The main result is that for $N \to \infty$ the photon statistics are rigorously Poisson on the lower branch, but on the upper branch quantum fluctuations persist and in the hysteresis region the statistics approach Bose–Einstein.

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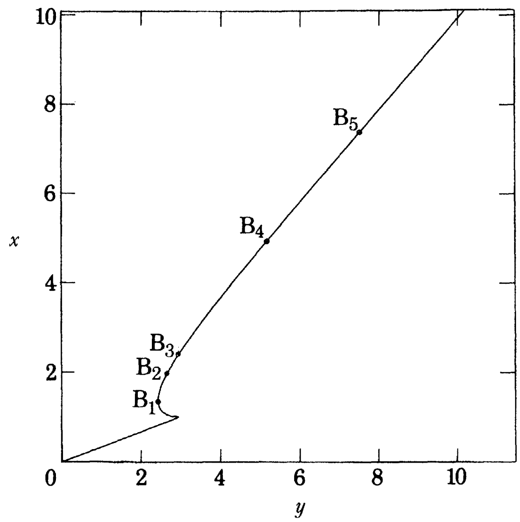

FIGURE 1. Plot of the transmitted field x against the incident field y. The points B_1 to B_5 on the upper branch refer to figure 2.

The density operator ρ for the whole system satisfies

$$d\rho/dt = -i\{(H_0+H_1+H_2), \rho(t)\} + \Lambda_a \rho(t) + \Lambda_f \rho(t), \tag{1}$$

$$H_0 = \omega_s S_z + \omega_s a^\dagger a,$$

$$H_1 = -igaS_+ + iga^\dagger S_-,$$

$$H_2 = i\kappa(a^\dagger E(t) - aE^*(t))$$

(Puri *et al.* 1984). The collective N atom operators $S_\pm$, S_z satisfy angular momentum commutation relations, $E(t)$ is the external field, $a(a^\dagger)$ are single-resonant-mode cavity field-operators, the atomic resonance is ω_s, g is a coupling constant, $\Lambda_a \rho$ and $\Lambda_f \rho$ describe collective spontaneous emission and cavity damping respectively. Details are in Puri *et al.* (1984). For large enough N we find $y \equiv 2|E|g/\gamma_0 N$ (γ_0 is the A-coefficient) and $x \equiv 2|\alpha|g/\gamma_0 N$ ($\alpha \equiv \langle a \rangle$) satisfy the input–output relation

$$\begin{aligned} y &= (1+C)\,x + O(N^{-1}), & x \leqslant 1 \\ &= x[1+C\{1-(x^2-1)^{\frac{1}{2}}/x^2 \arcsin(x^{-1})\}] + O(N^{-1}), & x \geqslant 1. \end{aligned} \tag{2}$$

Figure 1 is a plot of output x against input y for $N \rightarrow \infty$ and $C = 2$; $C \equiv g^2\kappa^{-1}/\gamma_0$ is a co-operation number and, since $C = Ng^2\kappa^{-1}/N\gamma_0$, collective $N\gamma_0$ replaces γ_0. Note the cusp at switch-up ($x = 1$) and that there is true bistability since the negative slope region is unstable. To reach (2) we 'decorrelate' matter and field operators in the equation of motion for atomic operators. This should be good for $N \rightarrow \infty$ (see Puri *et al.* 1984 and references therein).

Despite the interest of super-radiant emission in the approach to the steady state (Raimond *et al.* 1982*a*, *b*; Hildred *et al.* 1984), here we calculate only $P_n(\infty)$, the probability of finding n photons in the cavity mode (output x) in the steady state $t \rightarrow \infty$. Since

$$P_n(\infty) = \mathrm{Tr}\,\rho(\infty)|n\rangle\langle n| \equiv \langle |n\rangle\langle n| \rangle$$

and

$$|n\rangle\langle n| = (n!)^{-1}\sum_{r=0}^{\infty}(-1)^r(r!)^{-1}(a^\dagger)^{r+n}(a)^{r+n}$$

(compare with Sarkar & Elgin 1984)

$$P_n(\infty) = (n!)^{-1}\sum_{r=0}^{\infty}(-1)^r(r!)^{-1}\langle(a^\dagger)^{r+n}(a)^{r+n}\rangle. \tag{3}$$

Since, in the steady state, $a = E+g\kappa^{-1}S_-$, $P_n(\infty)$ can be expressed in terms of $(|E|^2)^r$ and $\langle(S_+)^j(S_-)^l\rangle$. The latter have been calculated in the steady state already (Hassan *et al.* 1980 and references therein). So after some manipulation we can reach a form

$$P_n(\infty) = (|E|^2/n!)^n\sum_{r=0}^{\infty}\{(-1)^r/r!\}|E|^{2r}\sum_{v,w=0}^{r+n}\binom{r+n}{v}\binom{r+n}{w}\{\ldots\}_{v,w}, \tag{4a}$$

where $$\binom{r+n}{v} \equiv (r+n)!/v!(r+n-v)!$$

and $$\{\ldots\}_{v,w} \equiv \left[\frac{C}{\{1+C(1-(N+1)/D)\}}\right]^{v+w} D^{-1}\sum_{m=\max(v,w)}^{N}\left|\frac{g\alpha}{\gamma_0}\right|^{2(N-m)} H_{N,m}. \tag{4b}$$

This form is exact for $N\rightarrow\infty$ (given the decorrelation mentioned). Definitions and manipulative details can be deduced from Puri *et al.* (1984) and they will be published at length elsewhere. Here we shall use the results

$$(N+1)/D = \begin{cases} 0, & x\leqslant 1, \\ (x^2-1)^{\frac{1}{2}}/x^2 \arcsin(x^{-1}), & x\geqslant 1; \end{cases}$$

$$D^{-1}\sum_{m=\max(v,w)}^{N}(\tfrac{1}{2}Nx)^{2(N-m)}H_{N,m} = \begin{cases} 1, & x\leqslant 1 \\ \{1-(x^2-1)^{\frac{1}{2}}/x^2\arcsin(x^{-1})\}\,S, & x\geqslant 1, \end{cases}$$

with $$S = \sum_{m=0}^{\max(v-1,\,w-1)}(2/x)^{2m}\{(m!)^2/(2m+1)!\},\quad x\geqslant 1.$$

The latter form comes from Drummond (1980) and is strictly valid only for $v, w \ll \infty$.

This way we find on the lower branch, $x\leqslant 1$, that

$$P_n(\infty) = (n!)^{-1}(|E|/(1+C))^{2n}\exp\left[-\{|E|/(1+C)\}^2\right]. \tag{5}$$

This is a Poisson distribution with mean $\bar{n} = \{|E|/(1+C)\}^2$, and the output x is coherent. Note that $|E| = (N\gamma_0 g^{-1})(\frac{1}{2}y)$: formally, as $N\rightarrow\infty$, $|E|\rightarrow\infty$ so that y remains finite; this is a 'thermodynamic limit' needed to establish an actual phase transition (characterized by (2) for $N\rightarrow\infty$) at $x = 1$ (Hassan *et al.* 1980). Alternatively we can consider the limit $N\rightarrow\infty$ so that $Ng^2\kappa^{-1}$, $N\gamma_0$, C and $|E|$ all stay finite. In practice we can work with N small (say $N\approx 50$) without changing the o.b. curve (2) very much (cf. Puri *et al.* 1984). In the experiments on Rydberg atoms (Raimond *et al.* 1982*a*) $g\approx 3\times 10^5$ Hz, $\gamma_0\approx 50$ Hz, so $N\gamma_0 g^{-1}\approx 1$ even for $N\approx 10^4$.

On the upper branch

$$P_n(\infty) = |E|^{2n}(n!)^{-1}\sum_{r=0}^{\infty}(-1)^r(r!)^{-1}|E|^{2r}\sum_{v,w=0}^{r+n}\binom{r+n}{v}\binom{r+n}{w}\{\ldots\}, \tag{6}$$

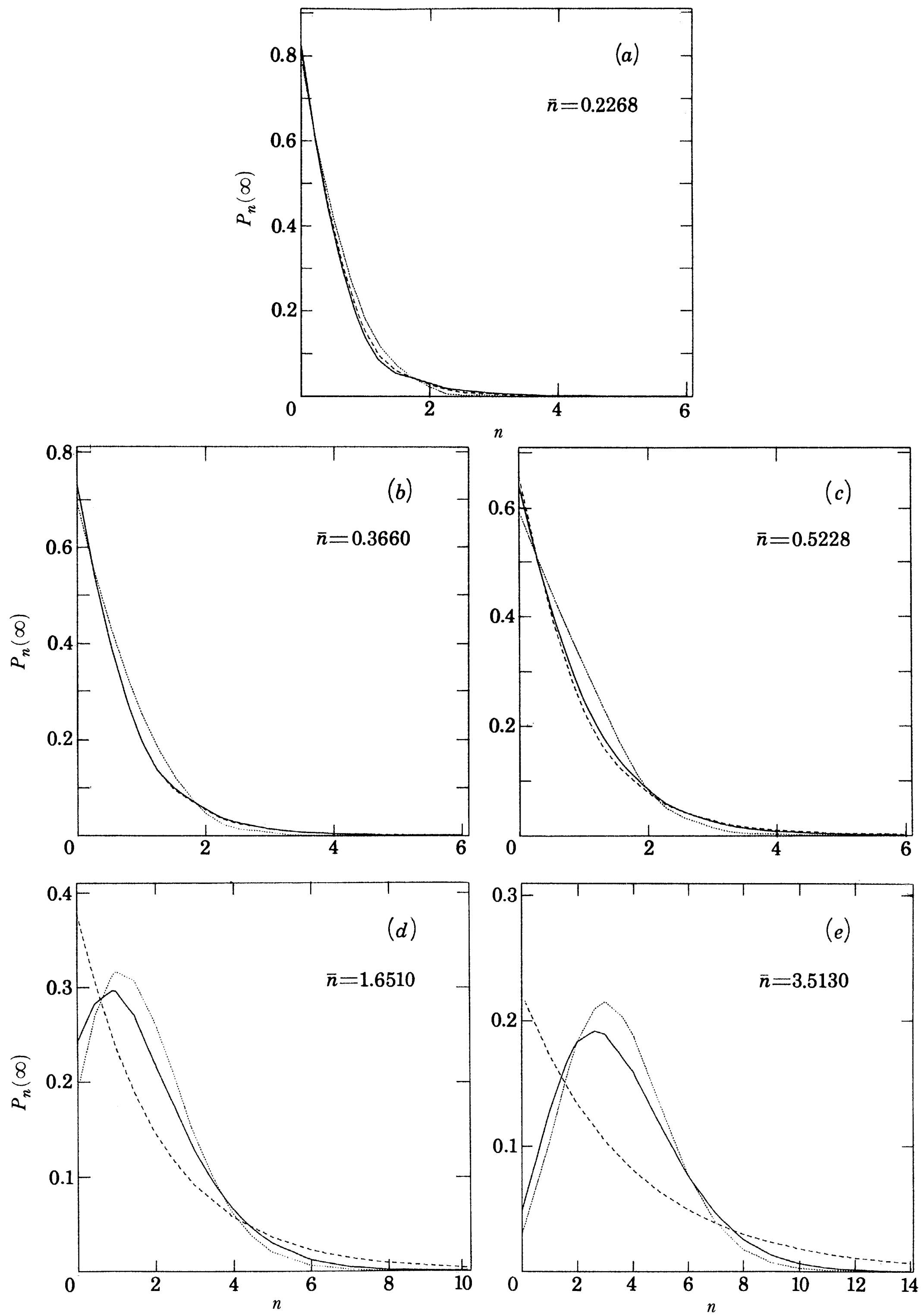

FIGURE 2. Plots of $P_n(\infty)$ (solid lines) against n for $C = 2$ corresponding, (*a*)–(*e*), to the points B_1 to B_5 on the upper branch of figure 1, respectively. The broken and dotted lines represent the Bose–Einstein and Poisson distributions for the same $\bar{n}$, respectively.

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where $$\{\ldots\} \equiv [\![-C/[1+C\{1-\sqrt{x^2-1}/x^2 \arcsin(x^{-1})\}]]\!]^{v+w}[\ldots] \tag{7a}$$

and $$[\ldots] \equiv [1-\{(x^2-1)^{\frac{1}{2}}/x^2 \arcsin(x^{-1})\} \sum_{m=0}^{\max(v-1,\,w-1)} (2/x)^{2m} m!/(2m+1)!]. \tag{7b}$$

This has no simple analytical form, so it is plotted for $C = 2$ and $N\gamma_0 g^{-1} = 0.5$ in figure 2*a*–*e* for the points $B_1(y = 2.439)$, $B_2(y = 2.632)$, $B_3(y = 3.0)$, $B_4(y = 5.15)$ and $B_5(y = 7.503)$ on the upper branch of the o.b. curve (figure 1). The solid lines are $P_n(\infty)$ (from (6)) against n; the Poisson (dotted lines) and Bose–Einstein (broken lines) distributions for the same $\bar{n}$ are also plotted. At B_1 (switch-down) $P_n(\infty)$ is evidently near Bose–Einstein, remains so beyond B_3 (switch-up), but tends to Poisson for large input fields. The calculated $g^{(2)}(0)$ are 2.508, 1.869, 1.598, 1.196 and 1.093, respectively: the first is particularly interesting because it exceeds 2.0 (the Bose–Einstein value). For $C = 0.1$ and $N\gamma_0 g^{-1} = 1.0$ we find $g^{(2)}(0) = 1.0012$, 1.0018, 1.0023, 1.0002 and 1.0001, so the low-cooperation number means the curves coincide with a Poisson distribution everywhere. Increasing C has an opposite effect.

These results give a measure of quantum fluctuations in the optically bistable region. The recent experiments on Rydberg atoms suggest they might be observable. It is open what significance such quantum fluctuations will have to real o.b. devices. In particular, their role in bifurcation to turbulence has also still be be determined, but they would presumably induce chaos earlier in any period multiplication sequence.

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